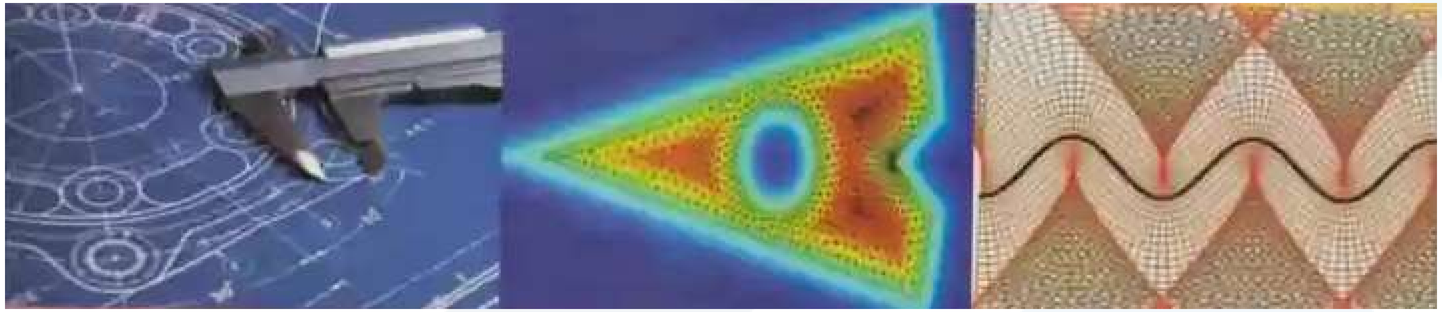
CEDC301: Engineering Mathematics

Lecture Notes 3: Integration in the Complex Plan



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Chapter 2

Integration in the Complex Plan

1. Contour Integrals
2. Cauchy-Goursat Theorem
3. Independence of the Path
4. Cauchy's Integral Formulas and Their Consequences

1. Contour Integrals

Terminology

Suppose C is a curve parameterized by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, and A and B are the points $(x(a), y(a))$ and $(x(b), y(b))$, respectively. We say that:

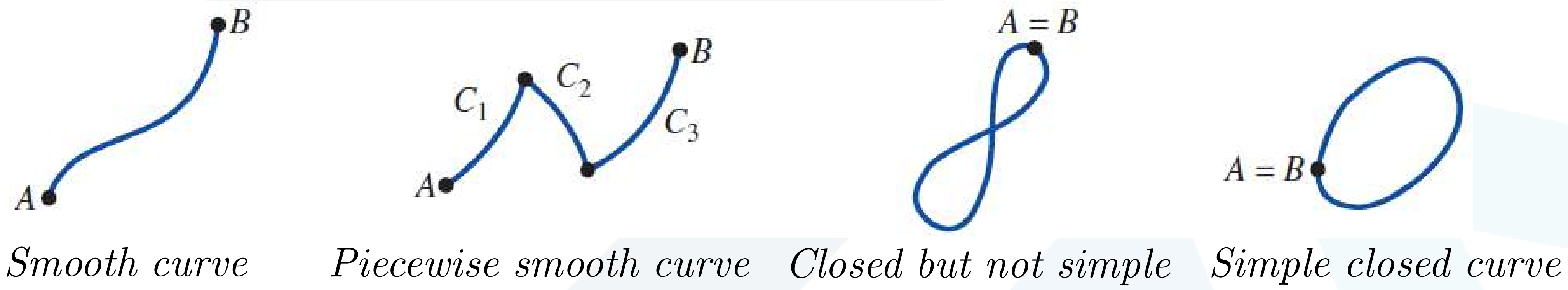
(i) C is a smooth curve if x' and y' are continuous on the closed interval $[a, b]$ and not simultaneously zero on the open interval (a, b).

(ii) C is piecewise smooth if it consists of a finite number of smooth curves $C_1, C_2, \ldots, C_n$ joined end to end, that is, the terminal point of one curve C_k coinciding with the initial point of the next curve C_{k+1}. $C = C_1 \cup C_2 \cup \ldots \cup C_n$.

(iii) C is a simple curve if the curve C does not cross itself except possibly at $t = a$ and $t = b$.

(iv) C is a closed curve if $A = B$.

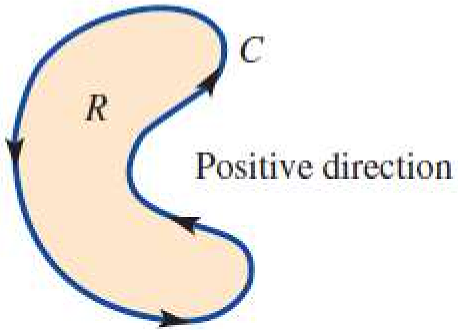
(v) C is a simple closed curve if $A = B$ and the curve does not cross itself ; that is, C is simple and closed.

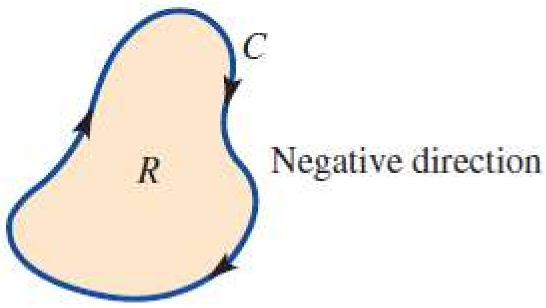


- Integration in the complex plane is defined in a manner similar to that of a line integral in the plane.
- Integral of a complex function $f(z)$ that is defined along a curve C in the complex plane. These curves are defined in terms of parametric equations $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, where t is a real parameter.

- By using $x(t)$ and $y(t)$ as real and imaginary parts, we can also describe a curve C in the complex plane by means of a complex-valued function of a real variable t: $z(t) = x(t) + iy(t)$, $a \leq t \leq b$.
- For example, $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$, describes a unit circle centered at the origin. This circle can also be described by $z(t) = \cos t + i \sin t$, or even more compactly by $z(t) = e^{it}$, $0 \leq t \leq 2\pi$.
- The point $z(a) = x(a) + iy(a)$ or $A = (x(a), y(a))$ is called the initial point of C and $z(b) = x(b) + iy(b)$ or $B = (x(b), y(b))$ is its terminal point.
- In complex variables, a piecewise-smooth curve C is also called a contour or path.
- If C is not a closed curve, then we say the positive direction on C (positive orientation), if we traverse C from its initial point A to its terminal point B.

- In other words, if C is described by $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then the positive direction on C corresponds to increasing values of the parameter t.
- In the case of a simple closed curve C, the positive direction roughly corresponds to the counterclockwise direction or the direction that a person must walk on C in order to keep the interior of C to the left.
- The negative direction on a contour C is the direction opposite the positive direction.
- If C has an orientation, the opposite curve, that is, a curve with opposite orientation, is denoted by $-C$.





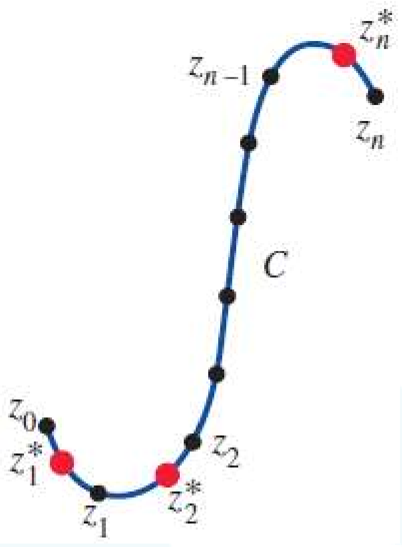
- On a simple closed curve, the negative direction corresponds to the clockwise direction.
- An integral of $f(z)$ on C is denoted by $\int_C f(z)dz$ or $\oint_C f(z)dz$ if the contour C is closed; it is referred to as a contour integral or simply as a complex integral.

Steps Leading to the Definition of the Complex Integral

1. Let $f(z) = u(x, y) + iv(x, y)$ be defined at all points on a smooth curve C defined by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$.
2. Divide C into n subarcs according to the partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$. The corresponding points on the curve C are: $z_0 = x_0 + iy_0 = x(t_0) + iy(t_0)$, $z_1 = x_1 + iy_1 = x(t_1) + iy(t_1)$, ..., $z_n = x_n + iy_n = x(t_n) + iy(t_n)$. Let $\Delta z_k = z_k - z_{k-1}$, $k = 1, \ldots, n$.
3. Let $\|P\|$ be the norm of the partition, i.e., the maximum value of $|\Delta z_k|$.

4. Choose a sample point $z_k^* = x_k^* + iy_k^*$ on each subarc.

5. Form the sum: $\sum_{k=1}^{n} f(z_k^*)\Delta z_k$



- **Definition:** Let f be defined at points of a smooth curve C defined by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$. The contour integral of f along C is

$$\int_C f(z)dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k$$

  The limit exists if f is continuous at all points on C and C is either smooth or piecewise smooth.

- **Theorem 1 (Evaluation of a Contour Integral):** If f is continuous on a smooth curve C given by $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

- **Example 1:** Evaluating a Contour Integral

Evaluate $\int_C \bar{z}\,dz$, where C is given by $x(t) = 3t,\ y(t) = t^2,\ -1 \le t \le 4$

$$\int_C \bar{z}\,dz = \int_{-1}^{4} (3t - it^2)(3 + 2it)\,dt = \int_{-1}^{4} (2t^3 + 9t)\,dt + i\int_{-1}^{4} 3t^2 dt = 195 + 65i$$

- **Example 2:** Evaluating a Contour Integral

Evaluate $\oint_C \frac{1}{z}\,dz$, where C is the circle $x(t) = \cos t,\ y(t) = \sin t,\ 0 \le t \le 2\pi$

$$\oint_C \frac{1}{z}\,dz = \int_0^{2\pi} (e^{-it})ie^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i$$

Properties

- **Theorem 2 (Properties of Contour Integrals):** Suppose f and g are continuous in a domain D and $C,\ C_1$ and C_2 are smooth curves lying entirely in D. Then

$$(i) \quad \int_C kf(z)dz = k\int_C f(z)dz,\ k \text{ a constant}$$

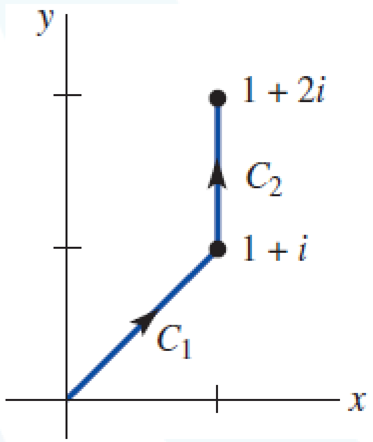
$$(ii) \quad \int_C [f(z)+g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$$

$$(iii) \quad \int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz,\ C = C_1 \cup C_2$$

$$(iv) \quad \int_{-C} f(z)dz = -\int_C f(z)dz$$

- **Note:** Theorem 2 also hold when C is a piecewise-smooth curve in D.
- **Example 3:** Evaluating a Contour Integral

Evaluate $\int_C (x^2 + iy^2)\,dz$, where C is the contour shown below



$$\int_C (x^2 + iy^2)\,dz = \int_{C_1} (x^2 + iy^2)\,dz + \int_{C_2} (x^2 + iy^2)\,dz$$

The curve C_1 is defined by $x(t) = y(t) = t$, $0 \le t \le 1$

The curve C_2 is defined by $x(t) = 1$, $y(t) = t$, $1 \le t \le 2$

$$\int_{C_1} (x^2 + iy^2)\,dz = \int_0^1 (t^2 + it^2)(1+i)\,dt = (1+i)^2 \int_0^1 t^2 dt = \frac{2}{3} i$$

$$\int_{C_2} (x^2 + iy^2)\,dz = \int_1^2 (1 + it^2) i dt = -\frac{7}{3} + i$$

$$\int_C (x^2 + iy^2)\,dz = \frac{2}{3} i - \frac{7}{3} + i = -\frac{7}{3} + \frac{5}{3} i$$

- Theorem 3 (A Bounding Theorem): If f is continuous on a smooth curve C and if $|f(z)| < M$ for all z on C, then $\left|\int_C f(z)dz\right| \le ML$, where L is the length of C.

$$L = \int_a^b \left|z'(t)\right| dt, \qquad z'(t) = x'(t) + iy'(t)$$

- Example 4: A Bound for a Contour Integral

  Find an upper bound for the absolute value of $\oint_C \frac{e^z}{z+1}\,dz$, where C is the circle $|z| = 4$.

  The length s of the circle of radius 4 is 8π. $|z+1| \geq |z| - 1 = 4 - 1 = 3$,

$$\left|\frac{e^z}{z+1}\right| \leq \frac{|e^z|}{|z|-1} = \frac{|e^z|}{3} = \frac{e^x}{3} \leq \frac{e^4}{3} \Rightarrow \left|\frac{e^z}{z+1}\right| \leq \frac{e^4}{3} \Rightarrow \left|\oint_C \frac{e^z}{z+1}\,dz\right| \leq \frac{8\pi e^4}{3}$$

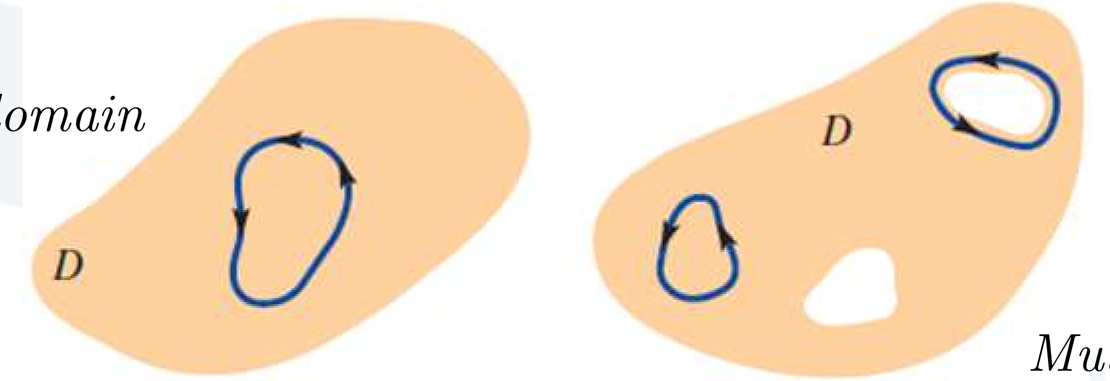
2. Cauchy-Goursat Theorem

Simply and Multiply Connected Domains

- A domain D is said to be simply connected if every simple closed contour C lying entirely in D can be shrunk to a point without leaving D.

- In other words, in a simply connected domain, every simple closed contour C lying entirely within it encloses only points of the domain D.
- A simply connected domain has no “holes” in it.
- The entire complex plane is an example of a simply connected domain.
- The annulus defined by $1 < |z| < 2$ is not simply connected.
- A domain that is not simply connected is called a multiply connected domain; that is, a multiply connected domain has “holes” in it.
- We call a domain with one “hole” doubly connected, a domain with two “holes” triply connected, and so on….
- The open disk defined by $|z| < 2$ is a simply connected domain;
- The open circular annulus defined by $1 < |z| < 2$ is a doubly connected domain.

Simply connected domain



Multiply connected domain

Cauchy's Theorem

Suppose that a function f is analytic in a simply connected domain D and that f' is continuous in D. Then for every simple closed contour C in D, $\oint_C f(z)dz = 0$

- Theorem 4 (Cauchy-Goursat Theorem): Suppose a function f is analytic in a simply connected domain D. Then for every simple closed contour C in D,

$$\oint_C f(z)dz = 0$$

- **Example 5:** The functions z^n with n a positive integer, $\sin z$, $\cos z$, e^z, $\sinh z$, and $\cosh z$ are analytic (they are entire functions), so for any closed contour C in the complex plane,

$$\oint_C z^n dz = \oint_C \sin z dz = \oint_C \cos z dz = \oint_C e^z dz = \oint_C \sinh z dz = \oint_C \cosh z dz = 0$$

- **Example 6:** Applying the Cauchy-Goursat Theorem

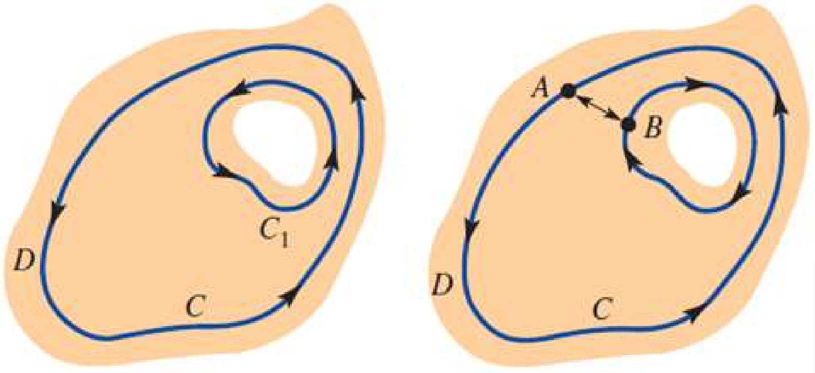
Evaluate $\oint_C \frac{1}{z^2} dz$, where C is the ellipse $(x-2)^2 + \frac{(y-5)^2}{4} = 1$

The rational function $f(z) = 1/z^2$ is analytic everywhere except at $z = 0$. But $z = 0$ is not a point interior to or on the contour C. Thus,

$$\oint_C \frac{1}{z^2} dz = 0$$

Cauchy-Goursat Theorem for Multiply Connected Domains

Suppose D is a doubly connected domain and C and C_1 are simple closed contours such that C_1 surrounds the “hole” in the domain and is interior to C. Suppose, also, that f is analytic on each contour and at each point interior to C but exterior to C_1.



When we introduce the cut AB the region bounded by the curves is simply connected.

The integral from A to B has the opposite value of the integral from B to A, so:

$$\oint_C f(z)dz + \int_{AB} f(z)dz + \int_{-AB} f(z)dz + \oint_{C_1} f(z)dz = 0 \Rightarrow \oint_C f(z)dz = \oint_{C_1} f(z)dz$$

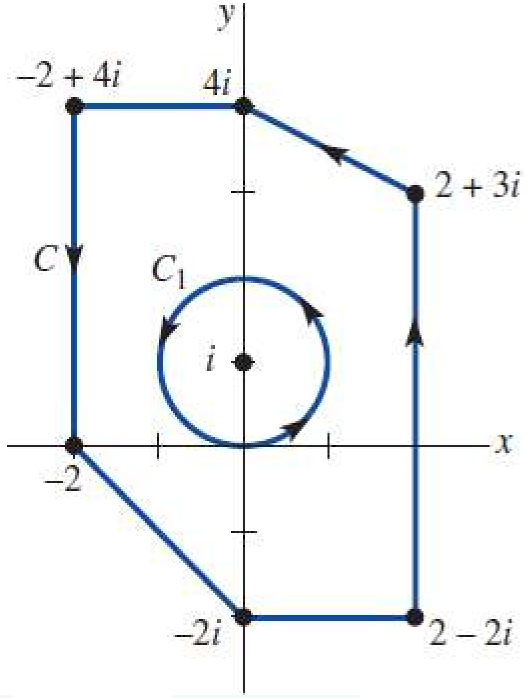
This result is sometimes called the principle of deformation of contours.

- Example 7: Applying Deformation of Contours

Evaluate $\oint_C \frac{1}{z-i}\,dz$, where C is the outer contour shown

We choose the more convenient circular contour C_1. By taking $r=1$, we are guaranteed that C_1 lies within C. C_1 is the circle $|z-i|=1$, which can be parameterized by $x=\cos t,\ y=1+\sin t$, or by $z=i+e^{it}$, $0\le t\le 2\pi$.

$$\oint_C \frac{1}{z-i}\,dz=\oint_{C_1}\frac{1}{z-i}\,dz=\int_0^{2\pi}\frac{ie^{it}}{e^{it}}\,dt=2\pi i$$



- If z_0 is any constant complex number interior to any simple closed contour C, then:

$$\oint_C \frac{dz}{(z-z_0)^n}=\begin{cases}2\pi i, & n=1\\ 0, & n \text{ an integer} \neq 1\end{cases} \qquad (*)$$

- Analyticity of the function f at all points within and on a simple closed contour C is sufficient to guarantee that $\oint_C f(z)dz = 0$.
- However, the result in $(*)$ emphasizes that analyticity is not necessary; in other words, it can happen that $\oint_C f(z)dz = 0$ without f being analytic within C.
- For instance, if C in Example 6 is the circle $|z| = 1$, then $(*)$, with the identifications $n = 2$ and $z_0 = 0$, immediately gives $\oint_C (1/z^2)dz = 0$.

  Note that $f(z) = 1/z^2$ is not analytic at $z = 0$ within C.

- Example 8: Applying Deformation of Contours

  Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}dz$, where C is the circle $|z-2| = 2$

  Since the denominator factors as $z^2 + 2z - 3 = (z-1)(z+3)$

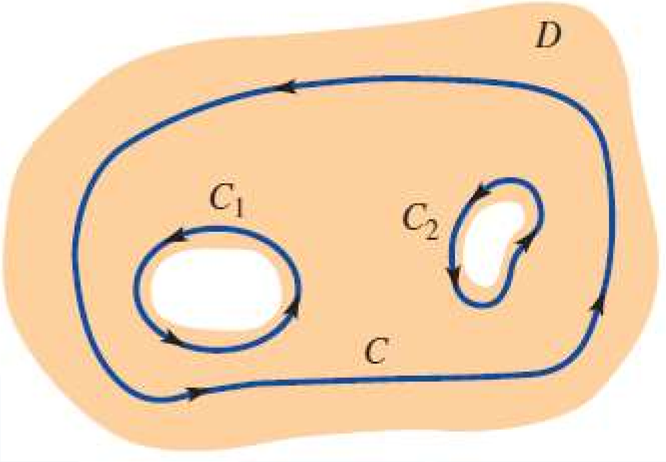
the integrand fails to be analytic at $z = 1$ and $z = -3$. Only $z = 1$ lies within the contour C, which is a circle centered at $z = 2$ of radius $r = 2$.

$$\frac{5z+7}{z^2+2z-3} = \frac{3}{z-1} + \frac{2}{z+3} \Rightarrow \oint_C \frac{5z+7}{z^2+2z-3}\,dz = 3\oint_C \frac{dz}{z-1} + 2\oint_C \frac{dz}{z+3}$$

$$\oint_C \frac{5z+7}{z^2+2z-3}\,dz = 3(2\pi i) + 2(0) = 6\pi i$$

- **Theorem 5 (Cauchy-Goursat Theorem for Multiply Connected Domains):** Suppose C, C_1, ..., C_n are simple closed curves with a positive orientation such that C_1, C_2, ..., C_n are interior to C but the regions interior to each C_k, $k = 1, 2, ..., n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, ..., n$, then:

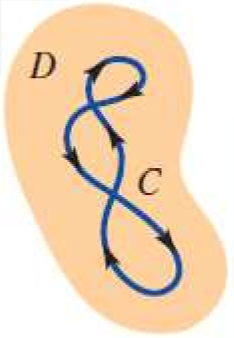
$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz$$



For example: triply connected domain D,

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$$

- Note: Cauchy-Goursat theorem is valid for any closed contour C in a simply connected domain D. As shown in the figure below.

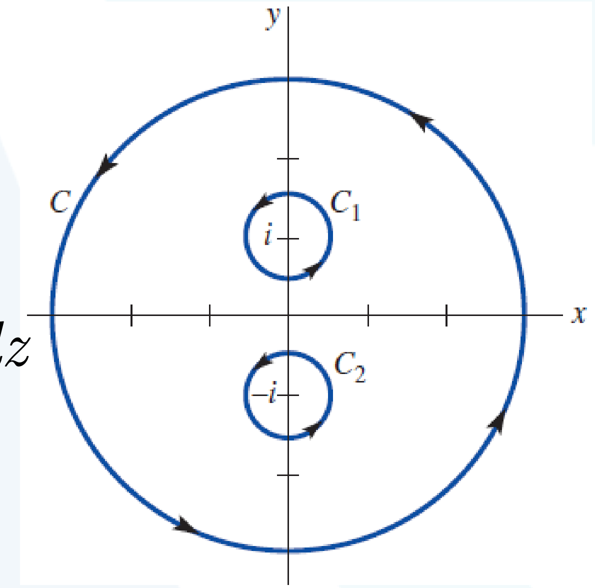


- Example 9: Applying Cauchy-Goursat Theorem for triply Connected domain

Evaluate $\oint_C \frac{dz}{z^2+1}$, where C is the circle $|z| = 3$

$z^2 + 1 = (z - i)(z + i)$, the integrand fails to be analytic at $z = i$ and $z = -i$. Both of these points lie within the contour C.

$$\frac{1}{z^2+1} = \frac{1/2i}{z-i} - \frac{1/2i}{z+i} \Rightarrow \oint_C \frac{dz}{z^2+1} = \frac{1}{2i}\oint_C \left[\frac{1}{z-i} - \frac{1}{z+i}\right]dz$$



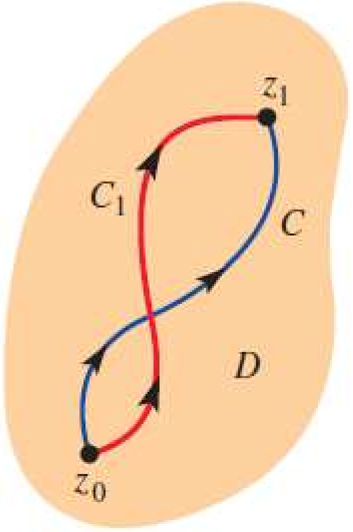
$$\oint_C \frac{dz}{z^2+1} = \frac{1}{2i}\oint_{C_1}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]dz + \frac{1}{2i}\oint_{C_2}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]dz$$

$$\oint_C \frac{dz}{z^2+1} = \frac{1}{2i}\left[(2\pi i) - (0)\right] + \frac{1}{2i}\left[(0) - (2\pi i)\right] = \pi - \pi = 0$$

3. Independence of the Path

- Definition: Let z_0 and z_1 be points in a domain D. A contour integral $\int_C f(z)dz$ is said to be independent of the path if its value is the same for all contours C in D with an initial point z_0 and a terminal point z_1.

  Suppose, that C and C_1 are two contours in a simply connected domain D, both with initial point z_0 and terminal point z_1. Note that C and $-C_1$ form a closed contour. Thus, if f is analytic in D, it follows from the Cauchy-Goursat theorem that

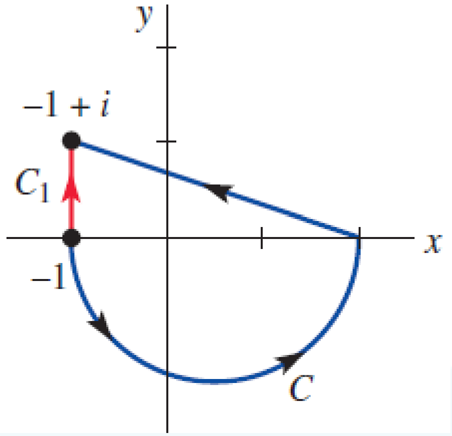


$$\oint_C f(z)dz + \oint_{-C_1} f(z)dz = 0 \Rightarrow \oint_C f(z)dz = \oint_{C_1} f(z)dz$$

- Theorem 6 (Analyticity Implies Path Independence): If f is an analytic function in a simply connected domain D, then $\int_C f(z)dz$ is independent of the path C.

- Example 10: Choosing a Different Path

Evaluate $\int_C 2z\,dz$, where C is the contour with initial point $z = -1$ and terminal point $z = -1 + i$ shown below



The function $f(z) = 2z$ is entire, we can replace the path C C_1 joining $z = -1$ and $z = -1 + i$. In particular, by choosing C_1 to be the straight line segment $x = -1$, $y = t$, $0 \le t \le 1$. $z = -1 + it$

$$\int_C 2z\,dz = \int_0^1 2(-1+it)i\,dt = -2i\int_0^1 dt - 2\int_0^1 t\,dt = -1-2i$$

- Definition: Suppose f is continuous in a domain D. If there exists a function F such that $F'(z) = f(z)$ for each z in D, then F is called an antiderivative of f.

For example, the function $F(z) = -\cos z$ is an antiderivative of $f(z) = \sin z$.

Antiderivative, or indefinite integral, of a function $f(z)$ is written

$$\int f(z)dz = F(z) + C$$

where $F'(z) = f(z)$ and C is some complex constant.

- **Theorem 7 (Fundamental Theorem for Contour Integrals):** Suppose f is continuous in a domain D and F is an antiderivative of f in D. Then for any contour C in D with initial point z_0 and terminal point z_1,

$$\int_C f(z)dz = F(z_1) - F(z_0)$$

- **Example 11:** Using an Antiderivative

$$\int_C 2zdz = \int_{-1}^{-1+i} 2zdz = z^2 \Big]_{-1}^{-1+i} = -1 - 2i$$

- Example 12: Using an Antiderivative

  Evaluate $\int_C \cos zdz$, where C is any contour with initial point $z = 0$ and terminal point $z = 2 + i$.

$$\int_C \cos zdz = \int_0^{2+i} \cos zdz = \sin z\Big]_0^{2+i} = \sin(2+i)$$

- If a continuous function f has an antiderivative F in a domain D, then $\int_C f(z)dz$ is independent of the path.
- If f is continuous and $\int_C f(z)dz$ is independent of the path in a domain D, then f has an antiderivative everywhere in D.
- Theorem 8 (Existence of an Antiderivative): If f is analytic in a simply connected domain D, then f has an antiderivative in D; that is, there exists a function F such that $F'(z) = f(z)$ for all z in D.

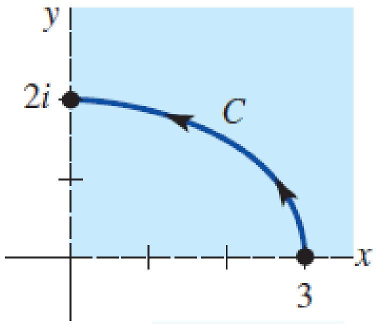
- Note: under some circumstances $\mathrm{Log}\ z$ is an antiderivative of $1/z$. For example, suppose D is the entire complex plane without the nonpositive real axis. The function $1/z$ is analytic in this multiply connected domain.

- If C is any simple closed contour containing origin, $\oint_C (1/z)dz = 2\pi i \neq 0$. In this case, $\mathrm{Log}\, z$ is not an antiderivative of $1/z$ in D, since $\mathrm{Log}\, z$ is not analytic in D.

- Example 13: Using the Logarithmic Function

Evaluate $\int_C \frac{dz}{z}$, where C is the contour shown below

Suppose that D is the simply connected domain defined by $x = Re(z) > 0$, $y = Im(z) > 0$. In this case, $\mathrm{Log}\, z$ is an antiderivative of $1/z$, since both these functions are analytic in D.



$$\int_C \frac{dz}{z} = \int_3^{2i} \frac{1}{z}\,dz = \text{Log } z\Big]_3^{2i} = \text{Log } 2i - \text{Log } 3 = \text{Ln } 2 + \frac{\pi}{2} i - \text{Ln } 3 = \text{Ln } \frac{2}{3} + \frac{\pi}{2} i$$

- **Example 14:** Using an Antiderivative of $z^{-1/2}$

Evaluate $\int_C \frac{1}{z^{1/2}}\,dz$, where C is the line segment between $z_0 = i$ and $z_1 = 9$

We take $f_1(z) = 1/z^{1/2}$ to be the principal branch of the square root function. In the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, the function $f_1(z) = 1/z^{1/2} = z^{-1/2}$ is analytic and possesses the antiderivative $F(z) = 2z^{1/2}$. Hence,

$$\int_i^9 \frac{1}{z^{1/2}}\,dz = 2z^{1/2}\Big|_i^9 = 2\left[3 - \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\right] = (6 - \sqrt{2}) - i\sqrt{2}$$

4. Cauchy's Integral Formulas and Their Consequences

We are going to examine several more consequences of the Cauchy-Goursat theorem. The most significant of these is the following result:

- The value of an analytic function f at any point z_0 in a simply connected domain can be represented by a contour integral.
- An analytic function f in a simply connected domain possesses derivatives of all orders.
- **Theorem 9 (Cauchy's Integral Formula or First Formula):** Let f be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If z_0 is any point within C, then:

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

- Example 15: Using Cauchy's Integral Formula

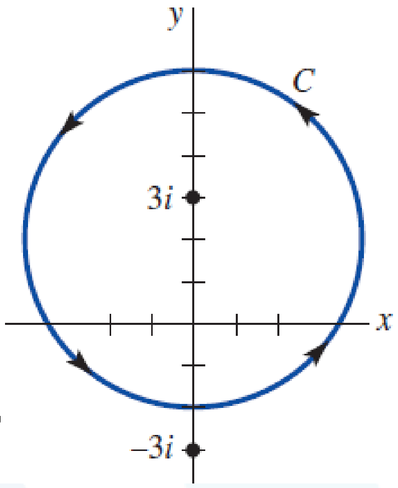
Evaluate $\oint_C \frac{z^2 - 4z + 4}{z + i}\, dz$, where C is the circle $|z| = 2$

$f(z) = z^2 - 4z + 4$ and $z_0 = -i$ as a point within the circle C. f is analytic at all points within and on the contour C.

$$\oint_C \frac{z^2 - 4z + 4}{z + i}\, dz = 2\pi i f(-i) = 2\pi i(3 + 4i) = 2\pi(-4 + 3i)$$

- Example 16: Using Cauchy's Integral Formula

Evaluate $\oint_C \frac{z}{z^2 + 9}\, dz$, where C is the circle $|z - 2i| = 4$



$$\frac{z}{z^2 + 9} = \frac{z/(z + 3i)}{z - 3i}$$

$z_0 = 3i$ is the only point within the circle C.

$f(z) = z/(z - 3i)$. This function is analytic at all points within and on the contour C.

$$\oint_C \frac{z}{z^2+9}\,dz = 2\pi i f(3i) = 2\pi i \frac{3i}{6i} = \pi i$$

- **Theorem 10 (Cauchy's Integral Formula for Derivatives or Second Formula):** Let f be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If z_0 is any point within C, then:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

- **Example 17:** Using Cauchy's Integral Formula for Derivatives

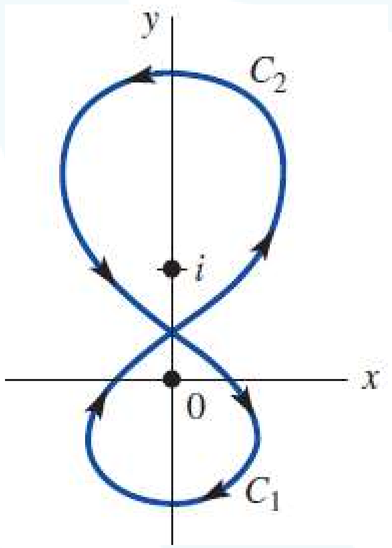
Evaluate $\oint_C \frac{z+1}{z^4+4z^3}\,dz$, where C is the circle $|z| = 1$

The integrand is not analytic at $z = 0$ and $z = -4$, but only $z = 0$ lies within the closed contour.

$$\frac{z+1}{z^4+4z^3} = \frac{(z+1)/(z+4)}{z^3} \Rightarrow \oint_C \frac{z+1}{z^4+4z^3}\,dz = \frac{2\pi i}{2!} f''(0) = \frac{3\pi}{32} i$$

- Example 17: Using Cauchy's Integral Formula for Derivatives

Evaluate $\oint_C \frac{z^3+3}{z(z-i)^2}\,dz$, where C is the contour shown below



C is not a simple closed contour, we can think of it as the union of two simple closed contours C_1 and C_2

$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}\,dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}\,dz$$

$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = -\oint_{C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z}\,dz + \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}\,dz = -I_1 + I_2$$

$$I_1 = \oint_{C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z}\,dz = 2\pi i f(0) = -6\pi i$$

$$I_2 = \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}\,dz = \frac{2\pi i}{1!} f'(i) = 2\pi i(3+2i) = 2\pi(-2+3i)$$

$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = -I_1 + I_2 = 6\pi i + 2\pi(-2+3i) = 4\pi(-1+3i)$$

Some Consequences of the Integral Formulas

- **Theorem 11 (Derivative of an Analytic Function Is Analytic):** Suppose that f is analytic in a simply connected domain D. Then f possesses derivatives of all orders at every point z in D. The derivatives f', f'', f''', ... are analytic functions in D.

- **Theorem 12 (Cauchy's Inequality):** Suppose that f is analytic in a simply connected domain D. and C a circle defined by $|z - z_0| = r$ that lies entirely in D. If $|f(z)| \le M$ for all points z on C, then:

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M}{r^n}$$

- **Theorem 13 (Liouville's Theorem):** The only bounded entire functions are constants.

- Theorem 14 (Fundamental Theorem of Algebra): If $p(z)$ is a nonconstant polynomial, then the equation $p(z) = 0$ has at least one root.
- Theorem 15 (Morera's Theorem): Suppose that f is continuous in a simply connected domain D and if $\oint_C f(z)dz = 0$ for every closed contour C in D, then f is analytic in D.
- Theorem 16 (Maximum Modulus Theorem for Analytic Functions): Suppose that f is analytic and nonconstant on a closed region R bounded by a simple closed curve C. Then the modulus $|f(z)|$ attains its maximum on C.
- If the stipulation that $f(z) \neq 0$ for all z in R is added to the hypotheses of Theorem 16, then the modulus $|f(z)|$ also attains its minimum on C.
- Example 18: Maximum Modulus

  Find the maximum modulus of $f(z) = 2z + 5i$ on the closed circular region $|z| \leq 2$.

$$|2z+5i|^2 = (2z+5i)\overline{(2z+5i)} = (2z+5i)(2\bar{z}-5i) = 4z\bar{z} - 10i(z-\bar{z}) + 25$$

$$|2z+5i|^2 = 4|z|^2 + 20\,Im(z) + 25$$

Because f is a polynomial, it is analytic on the region defined by $|z| \leq 2$. $\max_{|z|\leq 2}|2z+5i|$ occurs on the boundary $|z| = 2 \Rightarrow |f(z)|$ attains its maximum when $Im(z)$ attains its maximum on $|z| = 2$, namely, at the point $z = 2i$.

$$max_{|z|\leq 2}|2z+5i| = \sqrt{4(2)^2 + 20(2) + 25} = 9$$

- Note: In Example 18 $f(z) = 0$ only at $z = -5/2i$ and that this point is outside the region defined by $|z| \leq 2$. Hence $|f(z)|$ attains its minimum when $Im(z)$ attains its minimum on $|z| = 2$ at $z = -2i$.

$$min_{|z|\leq 2}|2z+5i| = \sqrt{4(2)^2 - 20(2) + 25} = 1$$